2. BREMSSTRAHLUNG RADIATION

In general, when discussing the properties of emission processes, we need to discuss basically the following aspects: 

a) the total emitted power by the single charge; 
b) the spectrum of radiation from a single charge; 
c) the radiation spectrum from an ensemble of particles; 
d) radiation self-absorption; 
e) polarization properties; 
f) the evolution of the particle energy spectrum because of the emission.

2.1 The classical limit. Small angle scattering. The electric dipole contribution.

We will discuss first the phenomenon in the classical limit, in spite that sometimes the photon energies will be comparable to those of interacting particles. So we will provide the results in the classical limit, which are the useful ones in most cases as they cover the relevant domain of the parameter space, and then we will discuss correction factors of quantum origin (see discussion about the Gaunt factor). Another condition is to neglect, on the first instance, a relativistic treatment, that will be discussed later in the chapter (less often we find astrophysical situations involving relativistic Bremsstrahlung).

The Bremsstrahlung process \(^1\) involves radiation emission from particle acceleration by Coulomb interaction with another particle. Now let us consider the main

\(^1\) The Bremsstrahlung process goes also under the name of free-free emission, because it involves energy levels of free particles. Bremsstrahlung means radiation from an accelerated particle.
contributions to the radiation field, because the situation for a realistic astrophysical plasma may be quite complicated in general. From what we have seen in Sect. 1.4.3, radiation from interaction of identical particles ($e^- \rightarrow e^-$, $p^+ \rightarrow p^+$) is null in the dipole approximation, while the quadrupole contribution, though different from zero, makes negligible radiant energy. Interactions of more than two particles can generate dipole and quadrupole terms depending on the kind of particles, but are very unlikely and rare, so they do not contribute significantly to the radiation field from astrophysical plasmas. So the relevant contributions come from interactions of two particles with different charge, that is interactions of electrons and protons (or electrons and ions in general). Note that in this case the magnetic dipole moment is null (Appendix 1A), so we can just consider for this the electric dipole component here.

Since the mass of the ion is much larger than that of the electron, we can assume just the motion of the latter and the ion at rest. Let us assume also that the motion of the electron is along a line that corresponds to the assumption that the motion is not violently perturbed by the interaction (see figure above).

During the transit in the vicinity of the ion the electron accelerates because of the Coulomb force and emits a pulse of radiation according to the Larmor relation. Let us then analyze the spectrum of photons produced. In the dipole limit, the electric field of the radiation is

$$|\vec{E}_{rad}| = \frac{\vec{d}}{c^2 r} \sin \theta$$  \hspace{1cm} [2.1]$$

with the orientation of the electric field as shown in the following graph

Now we exploit the discussion developed in Sect. 1.6 and remember that the spectrum of the pulse is given by the Fourier transform
\[
\frac{dW}{d\omega} = \frac{8\pi}{3c^3} \left| \tilde{\alpha}(\omega) \right|^2 = \frac{8\pi}{3c^3} \omega^4 \left| \tilde{\alpha}(\omega) \right|^2 \quad [1.35]
\]
i.e. it is proportional to the square of the Fourier transform of the acceleration of the dipole moment. Following [1.34] we have

\[
-\omega^2 \tilde{\alpha}(\omega) = -\frac{e}{2\pi} \int_{-\infty}^{+\infty} \dot{v} \exp(i\omega t) dt .
\]

Let us resolve in a simplified way this relation and define now the collision time as \( \tau \equiv \frac{b}{v} \). For \( \omega \tau >> 1 \) the integrand oscillates rapidly and the integral is small, the contrary for \( \omega \tau << 1 \), the exponential converges to 1, and the solution becomes

\[
-\tilde{\alpha}(\omega) = \begin{cases} 
\frac{e}{2\pi \omega^2} \Delta v & \text{for } \omega \tau << 1 \\
0 & \text{for } \omega \tau >> 1
\end{cases}
\]

where \( \Delta v \) is the velocity impulse that the electron acquires when passing close by the ion. The emitted power is

\[
\frac{dW}{d\omega} = \begin{cases} 
\frac{2e^2}{3\pi c^3} |\Delta v|^2 & \text{for } \omega \tau << 1 \\
0 & \text{for } \omega \tau >> 1
\end{cases} \quad [2.2]
\]

In our assumed case of a small perturbation of the electron orbit, \( \Delta v \) is easily calculated by integrating the acceleration from the Coulomb law. Assuming \( Z \) as the atomic number of the ion and the origin of times \( t=0 \) when the particle gets at the closest distance from the ion, and considering only the acceleration component orthogonal to the unperturbed trajectory as in the figure,
\[
 a_\perp = \frac{1}{m} \frac{Ze^2}{r^2} \sin \alpha = \frac{1}{m} \frac{Ze^2}{b^2 + v^2t^2} \frac{b}{\sqrt{b^2 + v^2t^2}}
\]

we get

\[
 \Delta v = \int_{-\infty}^{+\infty} a_\perp dt = \frac{Ze^2}{m} \int_{-\infty}^{+\infty} \frac{b dt}{(b^2 + v^2t^2)^{3/2}} = \frac{2Ze^2}{mbv}
\]  

having changed variable from \( t \) to \( x = bt/v \) (the integral becomes just a factor 2).

As a function of the impact parameter \( b \), the emitted spectrum of the pulse is

\[
 \frac{dW(b)}{d\omega} = \begin{cases} 
 \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2} & \text{for } b \ll v/\omega, \quad \omega \ll v/b \\
 0 & \text{for } b \gg v/\omega, \quad \omega \gg v/b 
\end{cases}
\]  

This expression does not give us the details of the function, but just its asymptotic behavior. An important point to note here is the independence of the pulse spectrum on frequency \( \omega \), a result that will imply a similar general property of Bremsstrahlung (both thermal and non-thermal). To note also the (inverse quadratic) dependences on mass, impact parameter – that are rather obvious – and that on velocity, this latter coming from the duration of the impact, that is very short when the velocity is high).

The next task is to determine the flux from a set of particles in a uniform medium with electron density \( n_e \) and ion density \( n_i \) and for particles with a fixed velocity \( v \). The electron flux will be \( n_e v \) and the differential area \( 2\pi b db \). Per unit volume and time (because we have an infinite train of packets, see our discussion of the time dependence of the spectrum in Sect. 1.5), we will have

\[
 \frac{dW(v)}{d\omega dV dt} = n_e n_i 2\pi v \int_{b_{\min}}^{b_{\max}} \frac{dW(b)}{d\omega} b db,
\]

where the limits in \( b \) are the minimum and maximum effective impact parameters contributing significantly to the radiation field. In practice, the detailed knowledge of the \( W(b) \) function is not needed and knowledge of the asymptotic behavior is enough.

From [2.3] and [2.4]

\[
 \frac{dW}{d\omega dV dt} = \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \ln \left( \frac{b_{\max}}{b_{\min}} \right)
\]  

[2.5]
2.2 The Gaunt factor

The logarithmic term in [2.5] is an important one because it includes all correction factors with respect to our so far simplified treatment, including the quantum corrections terms. Let us see it in some detail.

The $b_{\text{max}}$ term is that at which the asymptotic result for $dW/d\omega$ is no more valid and the contribution to radiation is small. We can then obviously take

$$b_{\text{max}} = \frac{v}{\omega}.$$ 

As for the lower limit $b_{\text{min}}$, it has to account for two factors.

a) The first one is our approximation of a linear weakly perturbed trajectory, and this can be set by comparison of the Coulomb potential and kinetic energy:

$$m v^2 = \frac{Ze^2}{b} \Rightarrow b_{\text{min}} = \frac{Ze^2}{m v^2}.$$ 

b) The second factor comes from the consideration that if $b$ is too small, the scattering cannot be treated any more as a classical process. From the uncertainty principle, $\Delta x \Delta p \geq h$, putting $\Delta x \sim b$ and $\Delta p \sim mv$, we get

$$b_{\text{min}}^2 = \frac{h}{mv}.$$ 

When $b < b_{\text{min}}^2$ the process cannot be treated any more as a classical one, and the quantum corrections become important. When $b_{\text{min}}^1 >> b_{\text{min}}^2$, the classic treatment holds valid, since we can show that the number of scatterings with $b < b_{\text{min}}^1$ are numerically negligible. This happens when

$$\frac{Ze^2}{m v^2} >> \frac{h}{mv} \Rightarrow v << \frac{Ze^2}{h} \sim Z \times 10^7 \text{ cm/sec}$$

(corresponding to a temperature of $T \approx 1000K$). When $v >> \frac{Ze^2}{h}$, a good approximation can be obtained by putting $b_{\text{min}} = b_{\text{min}}^2$.

Note that, in any case, the ratio $\ln\left(\frac{b_{\text{max}}}{b_{\text{min}}}\right)$ falls under the logarithmic operator, so it cannot make a large effect on the calculation of Bremsstrahlung emission. Then we can express [2.5] as

$$\frac{dW}{d\omega dV dt} = \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 g_{ff}(v, \omega) \tag{2.6}$$
having defined
\[ g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) \]
as the free-free Gaunt factor, dependent on the electron energy and the frequency of the emitted photon.

### 2.3 Thermal Bremsstrahlung

From an astrophysical point of view, the most interesting case is that of the thermal Bremsstrahlung, that is when the particle velocity distribution is thermal. In such case, the probability that an electron has velocity in \( d^3 \vec{v} \) is given by the Maxwellian distribution
\[ dP \propto e^{-E/kT} d^3 \vec{v} = \exp(-mv^2 / kT) d^3 \vec{v} = 4\pi v^2 \exp(-mv^2 / kT) dv, \]
having assumed an isotropic distribution of velocities. If we now integrate [2.6] over velocities, we get
\[ \frac{dW(T, \omega)}{dV dt d\omega} = \int_{v_{\text{min}}}^{\infty} \frac{dW(v, \omega)}{dV dt d\omega} v^2 \exp(-mv^2 / kT) dv. \]
The integral has been confined to a minimal velocity \( v_{\text{min}} \) because we need to produce photons of energy \( h\nu \), which is made only by particles having \( 1/2 \, mv^2 \geq h\nu \), so that \( v_{\text{min}} = (2hv/m)^{1/2} \), and \( d\omega = 2\pi dv \). By inclusion of relation (2.6) it turns out that the relevant integral is perfectly analytic (denominator is solvable numerically), and we finally get:
\[ \frac{dW}{dV dt d\nu} = 2^5 \pi e^6 \left( \frac{2\pi}{3km} \right) n_e n_i T^{-1/2} Z^2 \exp(-h\nu/kT) \bar{g}_{ff} \quad [2.7] \]
or in the CGS system
\[ \frac{dW}{dV dt d\nu} \simeq 6.810^{-38} Z^2 n_e n_i T^{-1/2} \exp(-h\nu/kT) \bar{g}_{ff} \quad [\text{erg} / \text{cm}^3 / \text{s} / \text{Hz}] \]
where $\bar{G}_{ff}$ is the Gaunt factor averaged over velocities. We report in the figure below (taken from Ribicki & Lightman) various approximations for this quantity in different domains of the plane of photo frequency and plasma temperature.

![Graph showing numerical values of the Gaunt factor $\bar{G}_{ff}(\nu, T)$. Here the frequency variable is $u = 4.8 \times 10^{11} \nu / T$ and the temperature variable is $\gamma^2 = 1.58 \times 10^5 Z^2 / T$. (Taken from Karzas, W. and Latter, R. 1961, Astrophys. J. Suppl., 6, 167.)](image)

We see that for $10^{-4} < u < 1$, the Gaunt factor varies from 1 to 5 ($u = h\nu / kT$). For $u >> 1$ the Bremsstrahlung emissivity decays exponentially and the Gaunt factor, though large, is no more important.

Another representation of analytic expressions for the Gaunt factor in different regimes of the $\nu - T$ plane is reported in the figure below. Note that all what concerns values of $h\nu > kT$ is of limited interest because the Bremsstrahlung flux is exponentially cutoff there.
The frequency dependence of the Gaunt factor in the astrophysically interesting region (small angle) is rather weak, of order of $\bar{g}_{\text{ff}} \approx \nu^{-0.2}$. This dependence shows up in the figure of the optically thin Bremsstrahlung spectrum below.

![Figure](image)

**Figure.** Approximate analytic formulae for the gaunt factor $\bar{g}_{\text{ff}}(\nu, T)$ for thermal bremsstrahlung. Here $\bar{g}_{\text{ff}}$ is denoted by $\bar{G}$ and the energy unit $\text{Ry} = 13.6$ eV. (Taken from Novikov, I. D. and Thorne, K. S. 1973 in Black Holes, Les Houches, Eds. C. DeWitt and B. DeWitt, Gordon and Breach, New York.)

In the optically thin limit, the Bremsstrahlung spectrum can be represented as an almost flat exponentially cutoff function like in the figure (including Gaunt factor):
X-ray emission from the Centaurus cluster of galaxies detected by proportional counters on HEAO1-A2 experiment. Note that the spectrum looks like steep instead of flat as is suppose to be because units in Y-axis are not photon energy but number.

Simulated X-ray spectrum by a hot plasma with solar metal abundance.
More in general, if we have a non-isotropic non-Maxwellian distribution of particles, we need to integrate [2.6] over the actual distribution of velocities, with the appropriate Gaunt factors. In such cases we talk of *non-thermal Bremsstrahlung*, see below an application in Sect. 2.6.

Relevant examples of high-energy spectra of cosmic sources immediately identified as free-free emissions are reported in the above figures. In the previous two cases the sources are hot plasmas present in rich galaxy clusters (see Sect. 2.7 below). The latter is a more complicated case including plasma emission in a star-forming galaxy plus an AGN contribution at photon energies larger than 4 keV.

**Polarization properties of free-free emission.**

For thermal plasmas, like those considered here, the motions of the particles in the emitting region are completely random. Consequently, this free-free emission is completely un-polarized, a feature that can disentangle among different interpretations of X-ray observations of X-ray astrophysical sources (again, see Sect. 3.9).
2.4 Cooling of a plasma through Bremsstrahlung emission

If we integrate [2.7] over all frequencies, we have in the CGS system the bolometric free-free emissivity:

\[
\varepsilon_{ff} = \frac{dW}{dVdt} \approx 1.410^{-27} Z^2 n_e n_i T^{1/2} \bar{g}_B \left[ \frac{\text{erg}}{\text{cm}^3 \text{s}} \right]
\]

with \( \bar{g}_B \) the Gaunt factor averaged over frequencies. It comes out that \( 1.1 < \bar{g}_B < 1.5 \).

The rate of energy loss per unit volume for a plasma of solar abundance \( (Z\approx1.35) \) can then be approximated as

\[
\varepsilon_{ff} \approx 2.4 \times 10^{-27} n_e^2 T^{1/2} \left[ \frac{\text{erg}}{\text{cm}^3 \text{s}} \right]
\]

We can now compare [2.8] with the thermal energy density of the plasma. For this we assume to consider an astrophysical plasma dominated by hydrogen, such that \( n_e \sim n_i \):

\[
\rho_e = \frac{3}{2} kT \cdot n_{\text{tot}} = \frac{3}{2} kT \cdot (n_e + n_i) \approx 5 \times 10^{-16} n_e T
\]

and the free-free cooling time-scale is obtained:
In some more detail, how a hot plasma cools down can be well understood by considering the above figure and compare it to the spectrum of a synthetic source with hotter temperature (2 keV) shown previously. It is evident that the line emission becomes progressively more important at lower temperatures, while the continuum emission is dominant at high $T$. At high $T$ the cooling is via continuum emission, at lower $T$ via line emission. The divide line is $T = 2 \times 10^7 \text{ K}$, or $kT = 0.5 \text{ keV}$, above which free-free cooling dominates.

### 2.5 Radiative transfer and Bremsstrahlung self-absorption

It is useful to briefly discuss here the effects of free-free self-absorption, which gives us the occasion to mention generalities on the radiative transfer in a medium.

Let us first report the radiative transfer equation. When radiation crosses matter, part of the radiant energy will be added to the field, and part will be subtracted.

#### 2.5.1 Radiation intensity

Let us first define the radiation intensity (or surface brightness) as

$$I_\nu = \frac{dW}{dA dt d\Omega d\nu} = \frac{dE}{dA dt d\Omega d\nu}$$

i.e. the amount of energy flowing per unit area, time, solid angle and frequency. $I_\nu$ is known to be independent of the distance from the observer in a flat (Euclidean) universe if there is no relativistic motion between source and observer. This comes
immediately by considering two arbitrary surfaces at a given distance $R$, such that the
flowing energies across them are equal:

$$dE_1 = I_1^1 dA_1 dt_1 d\Omega_1 d\nu_1 = dE_2 = I_2^2 dA_2 dt_2 d\Omega_2 d\nu_2.\quad$$

Since frequencies and times do not change, and solid angles have the same scaling with $R$, $d\Omega_2 = dA_2 / R^2$ and $d\Omega_1 = dA_1 / R^2$, also the two intensities will be the same. The intensity and surface brightness is independent on source distance.

A different situation applies, of course, if there is motion between the source and the observer, or in the case of a curved expanding universe. As well known, in an expanding universe the radiation intensity scales with the cosmological redshift as $I_\nu \propto (1+z)^{-4}$.

### 2.5.2 The radiative transfer equation

Let us define the emission and absorption coefficients. The *emission coefficient* or emissivity is defined as the radiant energy produced per unit volume, time, solid angle and frequency

$$dW = dE = j_\nu dV dt d\Omega d\nu.$$

The *absorption coefficient* is instead defined as the (fractional) loss of energy when crossing a given layer of matter of depth $ds$:

$$dI_\nu = -\alpha_\nu I_\nu ds$$

with $\alpha_\nu = [cm^{-1}]$. This equation has a trivial solution as

$$I_\nu = I_{\nu0} \exp(-\int \alpha_\nu ds) = I_{\nu0} \exp(-\tau_\nu),\quad$$

where $\tau_\nu \equiv \int \alpha_\nu ds$ is the medium’s optical depth. The coefficient $\alpha_\nu$ is simply related to the particle cross section $\alpha_\nu = n_\sigma_\nu = \rho_\sigma_\nu / m$ with obvious meaning of the symbols. Note that under the term absorption we mean to include both the *true absorption* and *stimulated emission*, as in both cases the variation is proportional to the intensity $I_\nu$: the net effect on the radiation field may be positive or negative according to the dominance of true absorption or stimulated emission.

We can incorporate all effects on the radiation field in a formally concise expression

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \quad [2.11]$$

In general we define as *thermal radiation* that emitted by matter in thermal equilibrium. *Black body radiation* corresponds instead to a more restrictive
2.14 condition of perfect equilibrium between matter and radiation. In the former case it can be shown that we can define a source function dependent on matter temperature $T$ and photon frequency $\nu$ and independent on the geometry of the source, such that

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = B_\nu(T) = \frac{2h\nu^3}{c^2 \left(e^{h\nu/kT} - 1\right)} \quad \text{(Kirchhoff & Planck laws)} \quad [2.12]$$

In terms of this source function, the transfer equation can be written as

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu,$$  \hspace{1cm} [2.13]

For a black body radiation we have

$$\frac{dI_\nu}{d\tau_\nu} = 0 \Rightarrow I_\nu = S_\nu = B_\nu(T).$$

For a thermal radiation we can apply the Kirchhoff law and write, for the emissivity in the free-free case

$$j_{ff} = \frac{dW}{dV dt d\nu} \frac{1}{4\pi} = \alpha_{ff} B_\nu(T)$$

and inverting this one we get the free-free absorption coefficient, from [2.6 and 2.12]

$$\alpha_{ff} = \frac{4e^6}{3mhc} \left(\frac{2\pi}{3km}\right)^{1/2} n_en_i T^{-1/2} Z^2 \nu^{-3} (1 - e^{-h\nu/kT}) \overline{g}_{ff} \ [\text{cm}^{-1}]. \quad [2.14]$$

- For $h\nu \gg kT$ \hspace{1cm} $\alpha_\nu \propto \nu^{-3}$ (Wien region)
- For $h\nu \ll kT$ \hspace{1cm} $\alpha_\nu \propto \nu^{-2}$ (Reyligh-Jeans region)

The net important effect is that the free-free absorption gets larger at lower frequencies, while at high photon frequencies the medium is optically thin.

2.5.3 Formal solution of the transfer equation and the total free-free spectrum

Let us now address how the full spectrum of a thermal free-free source will look like. This implies resolving formally the radiative transfer equation. To this end we define $A \equiv I_\nu e^{\tau_\nu}$ and $B \equiv S_\nu e^{\tau_\nu}$. By multiplying [2.13] by $e^{\tau}$, we get
\[ e^\tau \frac{dI_v}{d\tau_v} = -A + B = e^\tau \frac{d\left(\frac{A}{e^\tau}\right)}{d\tau_v} = \]

\[ e^\tau \frac{1}{e^\tau} \frac{d(A)}{d\tau_v} + Ae^\tau \frac{d\left(\frac{1}{e^\tau}\right)}{d\tau_v} = \frac{d(A)}{d\tau_v} - Ae^\tau \frac{e^\tau}{e^{2\tau}} \]

\[ \Rightarrow \frac{dA}{d\tau_v} - A = -A + B \Rightarrow \frac{dA}{d\tau_v} = B \]

or \( A(\tau) = A(0) + \int_0^\tau B(\tau')d\tau' \), and substituting the terms \( A \) and \( B \)

\[ I_v(\tau_v) = I_{v0} \exp(-\tau_v) + \int_0^{\tau_v} \exp(\tau'_v - \tau_v)S_v(\tau_v)d\tau'_v \quad [2.15] \]

The physical interpretation of this is obvious: the intensity is partly extinguished by absorption in the path from \( \tau = 0 \) to \( \tau = \tau_v \), and is partly increased from the integrated contribution of the source function along the same path. Assuming that the source function is constant \( S_v = \text{const} \), then the solution gets

\[ I_v(\tau_v) = I_{v0} e^{-\tau_v} + S_v(1 - e^{-\tau_v}) = S_v + e^{-\tau_v} (I_{v0} - S_v) \]

which at large optical depths implies \( I_v \Rightarrow S_v \). So for \( \tau_v \gg 1 \):

\[ I_v = S_v = \frac{j_v}{\alpha_v} \]

For \( hv \ll kT \), \( j_v \approx \text{const} \), we have \( I_v \propto \nu^2 \) and the overall spectrum looks like in the following figure.

Here are two expressions for the free-free opacity. From [2.14] in the Wien region:

\[ \alpha_{ff} = 3.7 \times 10^8 \, n_e n_i T^{-1/2} Z^2 \nu^{-3} \bar{g}_{ff} \quad [\text{cm}^{-1}] \]

In the Reylight-Jeans regime:

\[ \alpha_{ff} = 0.018 \, n_e n_i T^{-3/2} Z^2 \nu^{-2} \bar{g}_{ff} \quad [\text{cm}^{-1}] \]

[2.15]
2.6 Relativistic Bremsstrahlung

It is worthwhile just to mention a completely different approach to treat the case in which the kinetic energies of particles are so large to become relativistic, as a consequence for example of a very high plasma temperature. This is based on the method of \textit{virtual quanta} (introduced by von Weizsacker & Williams in 1934). This is still valid in a non-quantum mechanical regime, although the complete treatment would require the full environment of the quantum electrodynamics.

We then consider the collision of an electron with an ion. In the current case, it is useful to reverse the reference system by considering the electron at rest and a relativistically moving ion, as shown in the above figure. We can easily show, based on the relativistic Lorentz transformation of the $E$ and $B$ fields, that the electric field generated by the ion onto the electron is an e.m. pulse travelling in a plane orthogonal to the direction of relative velocity. This radiation pulse is then Compton scattered by the electron and detected by the observer back in the ion frame (essentially two Lorentz transformations are needed).

The situation is illustrated in the following figure, where we show the two reference systems, $K'$ is that of the electron at rest and the moving ion $Z e$, $K$ is back to the ion frame.
In the ion frame, the E and B fields can be written as

\[
E_x = \frac{qx}{r^3} \quad E_y = \frac{qy}{r^3} \quad E_z = \frac{qz}{r^3} \\
B_x = 0 \quad B_y = 0 \quad B_z = 0 \tag{2.17}
\]

while the Lorentz transform implies that going into the electron frame (with obvious meaning of the symbols):

\[
\begin{align*}
\tilde{E}_x' &= \tilde{E}_x \quad \tilde{E}_y' = \tilde{E}_y \\
\tilde{E}_z' &= \gamma (\tilde{E}_z + \beta \times \tilde{B}) \\
\tilde{B}_x' &= \tilde{B}_x \\
\tilde{B}_y' &= \gamma (\tilde{B}_y + \beta \times \tilde{E}) \\
\tilde{B}_z' &= \gamma (\tilde{B}_z + \beta \times \tilde{E})
\end{align*}
\tag{2.18}
\]

We need to apply this transformation to the fields in (2.17) to get the fields in the electron frame. Considering the geometry sketched in the figure above, we have still in the K' frame \( r' = (\gamma^2 v^2 t^2 + b^2)^{3/2} \) and assuming the origin of times \( t \) at the time of transit to minimum distance, we get

\[
\begin{align*}
E_x' &= \frac{qx}{r'^3} = q \frac{\gamma vt}{r'^3} \\
E_y' &= \frac{qy y'}{r'^3} \\
E_z' &= \frac{qz z'}{r'^3} \\
B_x' &= 0 \\
B_y' &= -\frac{q\gamma \beta z'}{r'^3} \\
B_z' &= \frac{q\gamma \beta y'}{r'^3}
\end{align*}
\]
Perhaps interesting to note, these equations may well be derived from the Lienard-Wiechert potentials in [1.20] (see Ribicki & Lightman). If we now insert the Lorentz transform to the coordinates (only the x coordinate transforms), and assuming that the z coordinate is null (no loss of generality for this), we finally get in the $e^-$ frame

$$
E_x = -\frac{qv't\gamma}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \quad B_x = 0
$$

$$
E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \quad B_y = 0 \quad [2.19]
$$

$$
E_z = 0 \quad B_z = \beta E_y
$$

having neglected the primed symbols. In the ultra-relativistic case $v=c; \beta=1; B_z = E_y$, the $E_y$ and $B_z$ fields display a strong maximum for $t^* \leq t_0 = b/\gamma v = 1/\gamma \cdot \tau$, where $\tau$ is the timescale of the duration of the transit.

For $t=0$, [2.19] becomes $E_y = q\gamma/b^2$, perfectly consistent with our results in Appendix [B1.2]. The $E_x$ field has instead a shifted maximum, and in any case $E_x << E_y$. The detailed behavior of the fields seen by the electron are reported in the figure below and in the note 2.

---

2 The behavior of $E_y$ is immediately found from [2.19]. The maximum of $E_y$ is given by $\max[E_y] = q\gamma/b^2$, and the FWHM of the e.m. pulse is immediately found by setting the function equal to half this maximum:

$$
q\gamma b/\left[\gamma^2 v^2 t^2 + b^2\right]^{3/2} = q\gamma/2b^2, \quad t_0 = b/\sqrt{2}\gamma v.
$$

As for the maximum of $E_x$ it is calculated from the derivative of [2.19]. We can rewrite $E_x = -q\left[\gamma^2 v^2 t^2/(\gamma v t)^{2/3} + b^2/(\gamma v t)^{2/3}\right]^{3/2} = -q\left[(\gamma v t)^{4/3} + b^2/(\gamma v t)^{2/3}\right]^{3/2}$, such that the maximum corresponds to the minimum of the denominator, that we can get from the derivative w.r.t. $\gamma v t$ set to 0: $\gamma v t = b/\sqrt{2}$. The value of $E_x$ at the maximum is given by substituting this in [2.19]: $E_x = 2q/3^{2/3}b^2$.

2.18
To calculate the emitted spectrum of the e.m. pulse in the reference frame of the electron, we need to take the Fourier transform of the main component, the $E_y$ in [2.19], and express it in terms of the modified Bessel function $K_1$ of order 1, that is:

$$
\tilde{E}(\omega) = \tilde{E}_y = \frac{1}{2\pi} q\gamma b \int (\gamma^2 v^2 t^2 + b^2)^{-3/2} \exp(i\omega t) dt
$$

$$
= \frac{q}{\pi b v} K_1 \left( \frac{b\omega}{\gamma v} \right)
$$

[2.20]

This integral comes from a change of variable from time $t$ to $t' = t\gamma v/b \equiv x$, after having collected $b$ from parenthesis, thus obtaining that the integral above becomes

$$
\int_{-\infty}^{\infty} \exp(i\omega bx / \gamma v)(1 + x^2)^{-3/2} dx
$$

that can be solved in terms of the Bessel function.

The emitted spectrum in the reference frame of the electron then gets after [2.33 Sect.1], reintroducing the primed symbols,

$$
\frac{dW'}{d\omega' dA'} \left[ \text{erg / cm}^2 / \text{Hz} \right] = c |\tilde{E}(\omega)|^2 = \frac{c q^2}{\pi^2 b^2 v^2} \left( \frac{b'\omega'}{\gamma v} \right)^2 K_1^2 \left( \frac{b'\omega'}{\gamma v} \right)
$$

This is a very important and general result concerning the emission of relativistic particles. The spectrum of the pulse is cutoff at $\omega > \gamma v / b$, which is simply the reciprocal of the timescale of duration $t_0$ (see note 2).

In the limit in which the scattering process in $K'$ can be described via the classical Thomson cross section (i.e. $\mathcal{K}\omega' << mc^2$), in the ultra-relativistic case, because the scattering in $K'$ is symmetric $\omega = \gamma\omega'(1 + \beta \cos \theta') \rightarrow \omega = \gamma\omega'$, and considering that energies and frequencies transform in the same way from $K'$ to $K$, for the single pulse we have ($b'=b$ does not transform):

$$
\frac{dW(b)}{d\omega} = \frac{dW'}{d\omega'} = \sigma_T \frac{dW'}{d\omega' dA'} = \frac{8Z^2 e^6}{3\pi b^2 c^5 m^2} \left( \frac{b\omega}{\gamma^2 c} \right)^2 K_1^2 \left( \frac{b\omega}{\gamma^2 c} \right)
$$

Now we need to integrate over all $b$ values, but there is no general analytic solution to this. There exist however an asymptotic solution for low frequencies, $\omega << \gamma v / b_{\text{min}}$, where $b_{\text{min}} = \mathcal{K} / mc$ is the minimum impact parameter set by U.P. Then summing up now all interactions happening in the unit volume, which is achieved as we have done in Sect. 2.3 by replacing the velocity with a constant $c$, we finally get for a set of monochromatic electrons with energy $\gamma mc^2$ in the low-frequency limit:
\[
\frac{dW}{dVdt \ d\omega} = \frac{16Z^2 e^6}{3m^2 c^4} n_e n_i \ln \left( \frac{0.68 \gamma^2 c}{\omega b_{\text{min}}} \right).
\]

This is completely analogous to eq. [2.6] except for the replacement \( v \to c \). For \( \hbar \omega' \gg mc^2 \) the Klein-Nishina cross section (Sect. 6) in the electron frame has to be used instead of Thomson’s.

For a thermal distribution of hot relativistic electrons and after integration over frequencies, the volume emissivity becomes:

\[
\frac{dW}{dVdt} = 1.4 \times 10^{-27} Z^2 n_e n_i T^{1/2} \bar{g}_\beta (1 + 4.4 \times 10^{-10} T) \ [\text{erg} / \text{cm}^3 / \text{s}] \quad [2.21]
\]

By comparison with [2.8], it is emphasized here the incidence of the relativistic corrections, that become important as soon as the plasma temperature gets comparable to or in excess of \( T \geq 10^9 \, K \). This is in keeping with the notion about the rest mass-energy of electrons being 511 KeV, photons of that energy or greater imply relativistic corrections.

**Bremsstrahlung Gamma Rays from our Galaxy**

The observed widespread emission by the Galaxy in gamma rays (see figure below) is indeed partly interpreted as relativistic Bremsstrahlung by high energy cosmic ray electrons bombarding the diffuse Galactic interstellar medium.

**Gamma-rays from the Galaxy**

Gamma-ray emission is detected from our Galaxy which is thought to arise from relativistic Bremsstrahlung from high energy electrons.

The radiative energy is carried by photons with \( h\nu - E_e \) energies in the range 30-100MeV, suggesting many relativistic electrons with \( \gamma \sim 100 \)
Observed and predicted $\gamma$-ray spectra in different regions of the Galaxy. The Galactic regions are indicated in the panels. The observations are shown by vertical bars. The predicted contributions of $\pi$ decays, inverse Compton scattering (IC) and relativistic Bremsstrahlung (bremss), as well as the inferred extragalactic background component, are shown by different line symbols. [Strong et al. 2004].

The interpretation is that these ultra-relativistic galactic cosmic ray electrons collide with the general interstellar medium of the Galaxy and galactic clouds and emit relativistic Bremsstrahlung radiation. Other components of the Galactic spectrum are Inverse Compton emission by relativistic electrons and the neutral pion $\pi^0$ decay. Indeed, during collisions between high energy particles and nuclei of atoms and molecules of the ISM, pions of all charges, $\pi^+$, $\pi^0$ and $\pi^-$ are produced. The positive and negative pions decay into positive and negative muons and neutrinos, the muons
decaying in turn into positrons and electrons with relativistic energies. The latter make a contribution to the low energy electron spectrum and the predicted presence of positrons provides a direct test of the importance of the pion production mechanism in interstellar space. The neutral pions $\pi^0$ decay into two $\gamma$-rays. In proton–proton collisions, the cross-section for the production of a pair of high energy $\gamma$-rays is roughly the geometric size of the proton, $\sigma_\gamma \approx 10^{-24}$ cm$^2$. The spectrum of $\gamma$-rays produced in such collisions has a broad maximum centered about 1000 MeV and is a characteristic signature of the neutral pion decay process. The $\pi^0$ rest mass energy is 135 MeV, so that the observed photon peak at 1000 MeV produced by $\pi^0$ decay indicates average kinetic energies of these particles around 1 GeV.

2.7 Applications to free-free emissions by hot intra-cluster plasmas

We report in the following some applications of the theory of the Bremsstrahlung process in the previous Sects. to the observations of X-ray emissions from the cores of clusters of galaxies, a discovery dating back to the UHURU mission (1971), and confirmed by all subsequent X-ray space observatory missions (among others, the Einstein Observatory, Ariel V, EXOSAT, GINGA; SAX, XMM Newton, Chandra). X-ray sources associated to the cores of galaxy clusters are very numerous in the local Universe, making about half of the low-z X-ray source population, the other half being Active Galactic Nuclei (AGNs). At higher-z AGNs become quite more numerous, while clusters more rare. These huge X-ray emissions from clusters are due to the presence of enormous amounts of high-temperature plasma, emitting via optically thin free-free.

Examples of such sources at both low and high-redshifts are reported in the following figures.
CHANDRA image of the galaxy cluster XMM 1358+6245, z = 0.33

55 ks
2.7.1 Physical parameters of the hot intra-cluster plasma

X-ray observations provide us with extensive data on the hot intracluster plasmas (as well as on plasmas in many other astrophysical environments), sufficient to derive all relevant physical parameters.

First of all, X-ray imaging, as shown in the previous figures, allows us to measure the extent of the plasma distribution. In clusters of galaxies, the emission is peaked at the cluster barycenter and smoothly decreases towards the outer regions. However, we can consider the plasma to cover the whole cluster extent of about $\ell_X = 1 \text{ Mpc}$ radius. Assuming that the cosmic distance of the cluster is known from redshift measurements (these can be easily achieved from optical spectroscopy, or even from X-ray spectroscopy using e.g. the numerous atomic lines detected by X-ray spectrographs, as in the figures in Sects. 2.3), the volume $V$ occupied by the plasma is immediately obtained: $V = 4\pi (\ell_X)^3/3 = 4(310^{24})^3 \approx 10^{72+2} \text{ cm}^3$. Then we can express the total X-ray luminosity from free-free emission by the plasma as

$$L_X = V \varepsilon_{ff} \quad [2.25]$$

where the free-free emissivity is given by [2.9]:

$$\varepsilon_{ff}$$
\[ \varepsilon_{ff} \approx 2.4 \times 10^{-27} n_e^2 T^{1/2} \left[ \frac{\text{erg}}{\text{cm}^3 \text{s}} \right]. \]  

Now the plasma temperature \( T \) here is easily measured from X-ray spectroscopy, either looking at the shape of the continuum comparing it with the exponential cutoff of the free-free emission eq. [2.7], or even more precisely by considering the X-ray line ratios compared with theoretical predictions based on ionization equilibrium (see Sect. 4). Plasma temperatures of about \( T = 10^7 \div 10^8 \text{ K} \) come out to be typical.

In turn, \( L_X \) is immediately obtained from the bolometric X-ray flux measurement integrating over the cluster area, together with the cluster luminosity distance \( d_L \):

\[
4\pi f_X a^2 = L_X = V \varepsilon_{ff} \approx 2.4 \times 10^{-27} n_e^2 T^{1/2} \times 10^{74} \left[ \frac{\text{erg}}{\text{cm}^3 \text{s}} \right]
\]

Then this analysis brings us the so far unknown quantity, the plasma particle density \( n_e \left[ \text{cm}^{-3} \right] \). Densities as low as \( n_e \approx 10^{-3} \div 10^{-4} \text{ cm}^{-3} \) are indicated.

Finally, the total X-ray luminosity by optically-thin Bremsstrahlung from a plasma with these parameters can be checked to be:

\[
L_X = V \varepsilon_{ff} \approx 2.4 \times 10^{-27} \left( \frac{n_e}{2 \times 10^{-4}} \right)^2 \left( \frac{T_e}{10^8} \right)^{1/2} \times 10^{74} \approx 10^{44} \text{ erg} / \text{sec}
\]

a huge luminosity in spite of the low particle density, mostly because of the enormous volumes involved. The total baryonic mass in the cluster plasma is

\[
M_X = V n_e m_H \approx 10^{74} \frac{n_e}{2 \times 10^{-4}} \times 10^{-24-4} \approx 4 \times 10^{46} / 2 \times 10^{33} \approx 2 \times 10^{13} M_\odot
\]

or somewhat larger than the baryonic mass condensed in stars and galaxies in the cluster core (see also Sect. 4 below).

### 2.7.2 Free-free optical depth

Knowing the basic physical parameters of the intracluster plasmas, we can now immediately verify if our assumption about the optical thinness of the free-free
emission is correct. Based on the expression for the absorption coefficient in the Reylight-Jeans regime [2.16]

$$\alpha_{ff} \approx 0.018 \, n_e^2 \, T^{-3/2} \, Z^2 \, \nu^{-2} \, \bar{G}_{ff} \, \text{[cm}^{-1}]$$  \hspace{1cm} \text{[2.30]}

let us first consider what happens in the intra-cluster plasmas. We have:

$$\alpha_{ff} \approx 0.02 \left( \frac{n_e}{10^{-3}} \right)^2 \left( \frac{T_e}{10^8} \right)^{-3/2} \left( \frac{Z}{1.4} \right)^2 \left( \frac{\nu}{10^{18}} \right)^{-2} \times 10^{-6-12-36} \times 2$$

$$\approx 4 \times 10^{-56} \, \text{[cm}^{-1}]$$  \hspace{1cm} \text{[2.31]}

corresponding to an optical depth for free-free self-absorption of

$$\tau_{ff} = \int_0^l \alpha_{ff} \, dl \approx \alpha_{ff} \, l \approx 410^{-56} \times 310^{24} \approx 10^{-31}$$  \hspace{1cm} \text{[2.32]}

an enormously small number! (Note for comparison that the Thomson scattering optical depth by the same electrons is much larger, of the order of 1 to 10%, as we will see in Sect. 7).

We also finally note that there are many astrophysical environments within which the free-free depth can get much higher. Note the strong dependence of $\alpha_{ff}$ on both the plasma temperature, density and the photon frequency. All these get much different values for example in plasmas in HII galactic regions: much lower temperatures, lower frequencies and higher densities, such that, in the average interstellar medium in our Galaxy:

$$\alpha_{ff} \approx \left( \frac{n_e}{1 \, \text{cm}^{-3}} \right)^2 \left( \frac{T_e}{10^4} \right)^{-3/2} \left( \frac{Z}{1.4} \right)^2 \left( \frac{\nu}{10^9} \right)^{-2} \times 410^{-2-6-18} \approx 410^{-26} \, \text{[cm}^{-1}]$$

$$\tau_{ff} \approx \alpha_{ff} \cdot 10 \, \text{pc} \approx 410^{-26} \times 310^{19} \approx 10^{-6}$$

However, in the Orion nebula we have $n \approx 700$ particles per cm, and the optical depth becomes unity $\tau_{ff} = 1$ at 1 GHz.

### 2.7.3 Astrophysical and cosmological implications of hot intra-cluster plasmas

As a matter of fact, hot intra-cluster plasmas include an enormous amount of diffuse baryonic matter in clusters of galaxies, as discussed in Sect. 4. This makes more
baryons than condensed in stars and galaxies, and so it constitutes a fundamental component of the cosmos. An interesting aspect to consider about it is the thermodynamical history of this huge reservoir of thermal energy, when was it brought to such high temperatures. An important aspect is then to consider the cooling time of these plasmas. Based on [2.10] we have, in the optically thin limit which applies here:

\[
t_{ff} = \frac{\rho_e}{\epsilon_{ff}} \approx 2.7 \times 10^{11} \left( \frac{n_e}{10^{-4}} \right)^{-1} \left( \frac{T_e}{10^8} \right)^{1/2} \times 10^{4+4} \text{ [sec]}
\]

\[
\approx 10^{12} \text{ [yrs]} \gg t_H
\]

about a factor 100 larger than the Hubble time \((t_H \approx 1.4 \times 10^{10} \text{ yrs})\). The cooling time may become comparable to the Hubble time, or even slightly lower, in the inner core regions of cool clusters.

This is an important conclusion: the hot plasma keeps memory of all heating processes happened since the formation of the cluster, and is a kind of integrated record.

Another consideration is about the dependence of the X-ray free-free emissivity on the plasma density

\[
I_X \propto n_e^2.
\]

This makes a different spatial dependence than that of galaxies, for which

\[
I_O \propto n_{Gal} \propto n_e
\]

In conclusion, the X-ray surface brightness is much more peaked at the cluster center than the galaxy distribution observable in the optical, and this implies a much easier detectability of cluster emission in X-rays: X-rays make a well-defined smooth continuum emission while galaxies are an ensemble of point sources, and in addition the centrally peaked emission is easily detectable even in noisy maps.