Clausius’ Virial Dynamical Theory

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Chapter 1

Clausius’ Virial dynamical Theory

The Clausius’ Virial Dynamical Theory (TCV) proposed by Secco (2000, 2001, 2005, hereafter LS1, LS5), which is able to justify many features of the galaxies Fundamental Plane (FP) (e.g., the existence and the trend of the tilt), is based on the existence of a maximum in the Clausius Virial potential energy (CV) of a stellar component (B) when it is completely embedded inside a dark matter (DM) halo (D). The role of the Clausius’ potential energy is taken into account within the framework of a dynamical explanation to the FP. The analysis has been carried out by the powerful tool of tensor virial theorem (Chandrasekhar, 1969; Spitzer, 1969; Binney and Tremaine, 1987), extended to two-component systems (e.g., Limber, 1959; Brosche et al., 1983; Caimmi et al., 1984; Caimmi and Secco, 1992; Dantas et al., 2000; Caimmi, 2004). The outputs of this kind of models were summarized and compared with some observable scaling relations for pressure-supported ellipticals and, in general, for two-component virialized systems.

In this context, a special (virialized) configuration is identified, and its occurrence is interpreted as a physical reason for the existence of the FP for Early Type Galaxies (ETGs). Clausius’ virial energy is maximized by the configuration under discussion, and the related radius (tidal radius) is claimed by the author to work as a scale length induced by the tidal action of D on B. The above mentioned choice makes a further constraint among physical parameters, and allows to reproduce the exponents, A and B, of the FP. In addition, the tilt ($\alpha_i \simeq 0.2$) is linked to a fixed cosmological scenario, instead of different amount of DM from galaxy to galaxy or breaking the strict
homology. For assigned values of $B$ and $D$ mass, the special configuration allows two-dimensionality scale relations for each of the three quantities: $\sigma_o$, $I_e$, $r_e$.

From the analysis of the dynamic and thermodynamic properties, the author deduces that the above defined tidal radius works as a confinement for the stellar subsystem, similarly to the tidal radius induced on globular clusters by the hosting galaxy, as von Hoerner (1958) found for the Milky Way. The new result for galaxies appears as a general extension of the old one to the case of concentric structures. It could add further insight to the fact, that different kinds of astrophysical objects, with a completely different formation history, but subjected to a tidal potential - and then characterized by a tidal radius - lie on the same FP (Djorgovski, 1995; Burstein et al., 1997; Secco, 2003). It should be noted that a tidal radius induced by a tidal potential appears to the author very similar to the truncation, which King (1966) introduced ad hoc in his primordial models for ellipticals, extrapolating known data for globular clusters. In addition, the exponents, $A$ and $B$, and the parameter, $\alpha_t$, related to the FP, are found to depend only on the inner, universal DM distribution, where the slope must range inside $0 \div 1$, implying that other families of galaxies or, in general, astrophysical virialized objects, necessarily belong to a similar FP (as observed in the cosmic meta-plane defined by Burstein et al., 1997).

1.1 Main lines of CV theory

1.1.1 Looking for a special virial configuration

To introduce the problem in a general way, we start by considering the potential well of a given spherical virialized dark matter halo of mass $M_D$ and virial radius $a_D$, with a density radial profile as follows:

$$\rho(r) = \frac{\rho_o}{(r/r_o)^\gamma [1 + (r/r_o)^\alpha]^\chi}, \quad \chi = \frac{(\beta - \gamma)}{\alpha}$$

(1.1)

where $\rho_o$ and $r_o$ is its characteristic density and its scale radius, respectively. These kinds of profiles have already been introduced by Zhao (1996) and by Kravtsov et al.(1998) in order to generalize the universal profile proposed by Navarro, Frenk & White (hereafter, NFW) (Navarro et al.1996, Navarro et al. 1997) which is obtained from Eq.(1.1) as soon as $(\alpha = 1; \beta = 3; \gamma = 1; \chi = 2)$. Hereafter, we will name them Zhao profiles.
The question which arises is the following: \textit{Does a special virial configuration exist among the infinite number of a priori possible virial configurations which the luminous (Baryonic) component \((B)\) may assume inside the given dark one \((D)\)?}

1.1.2 Tensor virial formalism

In order to find the answer we need to use the tensor virial theorem extended to two components: \(D + B\) (Brosche et al. 1983; Caimmi et al. 1984; Caimmi & Secco, 1992). In fact, according to current \(SCDM\) or \(ΛCDM\) cosmologies, collapsed structures, originated from density perturbations at recombination epoch, are surrounded by massive (non baryonic) dark halos. Therefore, the usual formulation of the virial theorem has therefore to be rewritten for taking into account this fact (e.g., Limber, 1959; Chandrasekhar, 1969; Spitzer, 1969; Caimmi et al., 1984; Binney & Tremaine, 1987; Caimmi and Secco, 1992; Dantas et al., 2000; Caimmi, 2004).

The extension to two-component tensor virial theorem reads (see also, Caimmi et al., 1984):

\[
2(T_u)_{pq} + (\Omega_u)_{pq} + (V_{uv})_{pq} = 0 \quad ;
\]

where the indices \(p=1, 2, 3\) and \(q=1, 2, 3\), denote the tensor components, \(u\) and \(v\) denote the subsystem under consideration (\(B\) or \(D\)), \(T_{pq}\) is the kinetic-energy tensor, \(\Omega_{pq}\) is the self potential-energy tensor and \(V_{pq}\) is the tidal potential-energy tensor.

The kinetic-energy tensor \(T_{pq}\), includes all the velocity components along the coordinate axes \(x_p\) and \(x_q\) and can be split in the sum of two contributions, \(T_{pq} = (T_{sys})_{pq} + (T_{pec})_{pq}\), related to systematic (e.g., rotation, streaming, vorticity) and peculiar (chaotic) motions, respectively. In collisional fluids (e.g., gas clouds) the extremely short mean free path of particles necessarily implies isotropic velocity distribution, while in collisionless fluids (e.g., stellar systems) an extremely long mean free path allows anisotropic velocity distribution, i.e., anisotropic pressure which yield by itself alone to a flattened configuration.

The self potential-energy tensor \(\Omega_{pq}\) depends on the mass distribution of the related subsystem which, in turn, defines the gravitational potential. The formulation of the scalar virial theorem is obtained by summing the diagonal
components in Eq.(1.2):

\[ 2T_u + \Omega_u + V_{uv} = 0 \] (1.3)

The tidal potential-energy tensor \( V_{pq} \) depends on the mass distribution of both subsystems: directly, with regard to the one under consideration, and via the gravitational tidal potential, with regard to the other. In the limiting situation of a vanishing total mass of the latter the tidal potential-energy tensor also vanishes and the virial theorem in tensor and in scalar form reduces to the usual one-component formulation.

The Clausius' virial, or virial potential energy is defined (Caimmi & Secco, 1992),

\[ V_u = \Omega_u + V_{uv}; \quad u = B, D \] (1.4)

where:

\[ \Omega_u = \int \rho_u \sum_{r=1}^{3} x_r \frac{\partial \Phi_u}{\partial x_r} \, d\vec{x}_u; \quad V_{uv} = \int \rho_u \sum_{r=1}^{3} x_r \frac{\partial \Phi_v}{\partial x_r} \, d\vec{x}_u, \] (1.5)

\( \Phi_u \) and \( \Phi_v \) being the gravitational potentials due to \( u \)-matter and \( v \)-matter distributions, respectively. CV is the generalization of Clausius’ virial for one-component system of mass points defined as the sum of the mass point positions scalar the forces on them. If the forces are due to the self-gravitation it coincides with the potential energy of the system. In a two-component system it relates to the energy contribution due to the two active forces on the respective subsystem: the self-gravity and the tidal-gravity due to the other component. Generally speaking, the total potential energy of this subsystem turns out to be different from the related CV.

According to Newton’s 1st theorem it follows that the mass fraction of the dark outer component which enters in the \( B \) Clausius’ tensor is only that which exerts dynamic effects on \( B \).

1.1.3 Two-component linear model

Then we need to model the two components, see Fig.(1.1). Let us define linear approximation or more briefly the linear model the system made of two homothetic similar strata spheroids\(^1\) described by two power-law mass density profiles and two different homogeneous cores. The linear model rep-

\(^1\)For the sake of simplicity, we limit ourselves to the spherical case without losing the validity of spheroidal case, which may be recovered simply by introducing a form factor, \( F \), equal 2 in the spherical case, see, Eq.(1.9, 1.10)
1.1. MAIN LINES OF CV THEORY

Figure 1.1: A two-components model where a homeoidally striated baryonic component (B) is embedded within a homeoidally striated dark matter halo (D). The DM isopycnic (i.e., constant density) surface tangent to the B component at the top major axis, is denoted as $\Sigma^*$. The homeoid bounded by $\Sigma^*$ and the external surface of the D component, $\Sigma$, exerts no dynamical action on B, owing to the Newton’s first theorem. In the special case where B and D are, in turn, homothetic (as assumed in the text), $\Sigma^*$ coincides with the external surface of the B component (Raffaele, 2003).

represents, to a good extent, a baryonic (stellar) component embedded within a DM halo with smoothed density profiles, expressed as:

\[
\rho_D = \frac{\rho_{oD}}{1 + \left(\frac{r}{r_{oD}}\right)^a}, \quad C_D = \frac{a_D}{r_{oD}}^a
\]

\[
\rho_B = \frac{\rho_{oB}}{1 + \left(\frac{r}{r_{oB}}\right)^b}, \quad C_B = \frac{a_B}{r_{oB}}^b
\]

(1.6)

(1.7)

where $C_B$ and $C_D$ are the two concentrations of the two components. In earlier models, $r_{oB}$ and $r_{oD}$ were the radii of the two different homogeneous cores, which typically assumed one tenth of the virial radii, $a_B$ and $a_D$, respectively.
respectively. Accordingly, the concentrations in the smoothed profiles both become equal to ten. The density profiles defined by Eq.(1.6) and Eq.(1.7), have the advantage that they may be considered as a generalization of pseudo-isothermal profiles which, in turn, may be regarded as sub-cases of the more general \textit{Zhao profiles} when: $\gamma = 0; \beta = \alpha = b, d$.

But a realistic elliptical model has to be: e.g., a stellar component with a Hernquist (1990) (hereafter, Her) density profile and a dark halo with a \textit{cored} or \textit{cuspy} NFW profile (that means, according to Eq.(1.1), respectively: $\alpha = 1; \beta = 4; \gamma = 1; \chi = 3$ and $\alpha = 1; \beta = 3; \gamma = 0; \chi = 3$, for the \textit{cored} NFW, $\alpha = 1; \beta = 3; \gamma = 1; \chi = 2$, for the \textit{cuspy} NFW) (as in Marmo, 2003, where two homeoidally striated ellipsoids are considered).

Then, the problem of transferring the outputs obtained with two cored powerlaw profiles (which also hold, to a good extent, for the models with smoothed profiles Eq.(1.6,1.7)) to the more general class of models with \textit{Zhao profiles} given by Eq.(1.1), is still open, as we will see in the next chapter.

We aim to find an explanation to some scaling relations for elliptical galaxies, which are essentially relationships among the exponents of the three quantities:

- $r_e =$ the effective radius;
- $I_e = \frac{L}{2\pi r_e^2} =$ mean effective surface brightness within $r_e$;
- $\sigma_o =$ the central projected velocity dispersion.

The advantage of a simple cored power-law model is that it is able to extract, in a completely analytical way, a number of main correlations, highlighting the interplay of the parameters. Moreover, this preliminary analysis may also underline what are simply details in the model and what, on the contrary, is strictly connected with the physical reason for the existence of a FP for two-component virialized systems. That allows us to open the road for a generalization of present results.

1.1.4 Special virial configuration

For the sake of simplicity, the outer component $D$ is assumed as frozen; this constraint will not essentially influence our results in order to determine the main features of the dynamic evolution of ETGs (and of virialized structures in general). The main reason for this assumption is that the masses of the
two components are not equal, the outer one being about ten times the inner one. As a consequence, tidal influences acting from the inner to the outer one is weaker than the reverse (e.g., Caimmi and Secco, 1992; Caimmi, 1994).

If \( d \) and \( b \) are the exponents of the power-law profiles of mass density distributions for the \( D \) and \( B \) component respectively, the Clausius’ virial tensor trace is:

\[
V_B = \Omega_B + V_{BD} \tag{1.8}
\]

\[
\Omega_B = \int \rho_B \sum_{r=1}^{3} x_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_B = -\nu \Omega_B \frac{GM_B^2}{a_B} \mathcal{F} \tag{1.9}
\]

\[
V_{BD} = \int \rho_B \sum_{r=1}^{3} x_r \frac{\partial \Phi_D}{\partial x_r} \, d\vec{x}_B \simeq -\nu' V_{\Omega_B} GM_B \tilde{M}_D \tag{1.10}
\]

where \( \tilde{M}_D = M_D \left( \frac{a_B}{a_D} \right)^{3-d} \) is the \( D \) mass fraction which, according to the Newton’s 1\textsuperscript{st} theorem, exerts a dynamical effect on the \( B \) component, \( \Phi_B \) and \( \Phi_D \) are the gravitational potentials due to \( B \) and \( D \) components, \( M_u \) and \( a_u \) \((u = B, D)\) are respectively the mass and the major semi-axis of the two components and \( \mathcal{F} \) is a form factor. To a good extent it turns out to be \( \tilde{m} = \frac{\tilde{M}_D}{M_B} = m \left( \frac{a_B}{a_D} \right)^{3-d} \) where \( m \) is the dark to bright mass ratio. Then the total mass which exerts a dynamical effects on \( B \) is:

\[
M_{\text{dyn}} = M_B + \tilde{M}_D \tag{1.11}
\]

As we can see in Fig.(1.2), a special configuration appears due to the occurrence of a maximum (CVM) in the Clausius’ virial energy trend (and then a minimum in the kinetic energy), under the following constraints:

\[
0 \leq b < 3 \quad ; \quad 0 \leq d < 2 \Rightarrow 0 \leq (b + d) < 5 \; ; \tag{1.12}
\]

The \( B \) component at CVM is characterized by the following major semi-axis, hereafter quoted as tidal radius, \( a_t \):

\[
a_t = \left[ \nu_\Omega \frac{1}{\nu'} \left( \frac{M_B}{M_D} \right)^{2-d} \right]^{\frac{1}{3-d}} a_D \tag{1.13}
\]

The total potential energy \( (E_{\text{pot}})_B \), on the contrary, is always monotonic (see, Fig.1.2). Indeed, by definition:

\[
(E_{\text{pot}})_B = \Omega_B + W_{BD} \tag{1.14}
\]
Figure 1.2: The energy trends of the $B$ component as a function of size ratio $x = a_B/a_D$, normalized at the factor $(GM_B^2 F)/a_D$. The Clausius’ virial energy curves ($V_n$, $n=1, 2, 3, 4$) and the total potential energy curves ($E_n$), are represented for the 4 cases of Tab.2.1 (Raffaele, 2003). In all 4 cases the Clausius’ virial shows a maximum. The results are related to first models (LS1) but they hold also at the linear approximation in the case of smoothed profiles, Eqs.(1.6, 1.7).

where $W_{BD}$ is the interaction energy tensor trace defined as:

$$ (W_{BD})_{ij} = -\frac{1}{2} \int_B \rho_B(\Phi_D)_{ij} d\vec{x}_B \quad ; \quad (1.15) $$

The mathematical connection between the interaction energy $W_{BD}$ and the tidal energy $V_{BD}$ is given by:

$$ V_{BD} = W_{BD} + Q_{BD} \quad (1.16) $$

where $Q_{BD}$ is a residual antisymmetric energy quantity.

For this linear model the following relationships can be rewritten in order
1.1. MAIN LINES OF CV THEORY

Table 1.1: The values of $\nu_\Omega B$, $\nu_\Omega D$ and $\nu_V'$ and the value of the tidal radius $x_t = a_B/a_D$ for different power-law exponents of the density profiles of the two components (see text).

<table>
<thead>
<tr>
<th>cases</th>
<th>b</th>
<th>d</th>
<th>$x_t$</th>
<th>$\nu_\Omega B$</th>
<th>$\nu_\Omega D$</th>
<th>$\nu_V'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.389</td>
<td>0.300</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>2)</td>
<td>0.0</td>
<td>0.5</td>
<td>0.346</td>
<td>0.300</td>
<td>0.312</td>
<td>0.333</td>
</tr>
<tr>
<td>3)</td>
<td>0.5</td>
<td>0.5</td>
<td>0.361</td>
<td>0.312</td>
<td>0.312</td>
<td>0.313</td>
</tr>
<tr>
<td>4)</td>
<td>1.5</td>
<td>0.5</td>
<td>0.419</td>
<td>0.367</td>
<td>0.312</td>
<td>0.254</td>
</tr>
</tbody>
</table>

to obtain the Fundamental Plane (FP):

$$M_B = \sigma_o^2 r_e c_2 G_1$$  \hspace{1cm} (1.17)  

$$r_e = c_2 c_1^{-1} \sigma_o^2 I_e^{-1} G_2 \frac{L}{M_{dyn}}$$  \hspace{1cm} (1.18)  

$$G_2 = G_1 (1 + \bar{m}) = \frac{1 + \bar{m}}{F [\nu_\Omega B + \bar{m} \nu_V']}$$  \hspace{1cm} (1.19)  

If we now choose for the $B$ component the special virial configuration $\bar{m}$ and $G_2$ become, respectively:

$$\bar{m}_t = \frac{\nu_\Omega B}{\nu_V'} \frac{1}{2 - d}$$  \hspace{1cm} (1.20)  

$$G_{2t} = \frac{1}{F} \frac{\nu_V'(2 - d) + \nu_\Omega B}{\nu_\Omega B \nu_V'(3 - d)}$$  \hspace{1cm} (1.21)  

Then, if we consider strict homology+universality of DM profile, both are constant and the exponents $A$, $B$ become inexorably fixed at 2, -1. We need of one other physical ingredient which has to be able to change these exponents. That is the capability of equipartition between the self and tidal potential energy the CV maximum has. Then, without breaking virial equilibrium and loosing the homology, considering the $B$ component as pressure supported (i.e., peculiar kinetic energy $T_{pec}$ is dominant with respect to the rotational kinetic energy $T_{rot}$, $T_{pec} >> T_{rot}$) which means:

$$\frac{1}{2} M_B \langle \sigma^2 \rangle \simeq \left( -\frac{\Omega_B - V_{BD}}{2} \right)_{a_B=a_t} ; \quad T_{rot} << T_{pec} ;$$  \hspace{1cm} (1.22)  

(where $\langle \sigma^2 \rangle$ is the mean square velocity dispersion of the $B$ component), at the special configuration we obtain:

$$a_t \simeq \left( \frac{\frac{1}{2} M_B \frac{\sigma^2}{k_B} a_D^{3-d}}{\nu^2 M_B M_D \mathcal{F}} \right)^{\frac{1}{2-d}}; \quad (1.23)$$

The Eq.(1.6) yields the following FP:

$$r_e \sim \sigma_o^{\frac{2}{2-d}} a_D^{-\frac{d}{2-d}} m^{-\frac{1}{2-d}} M_B^{-\frac{1}{2-d}} ; \quad (1.24)$$

That means:

$$\begin{align*}
\sigma_o^A &= \sigma_o^{\frac{2}{2-d}} ; \\
I_e^B &= a_D^{\frac{3-d}{2-d}} m^{-\frac{1}{2-d}} M_B^{-\frac{1}{2-d}} ;
\end{align*} \quad (1.25)$$

To share in almost equal amounts tidal potential and self potential energy is the main property of the Clausius’ virial maximum. On this basis are grounded all the main scale relations which the dynamical theory is able to predict for ETGs. Indeed, from Eq.(1.25) we can derive all the outputs of the theory as we will see in the next sections.

### 1.2 Mechanical arguments

The tidal radius configuration satisfies the d’Alembert principle of virtual works and then it is an equilibrium configuration (see LS1). But due to the fact that the total potential energy of the $B$ component has not a minimum (see, Fig.1.2), the equilibrium is unstable.

By definition, the work $L_s$ done by the self gravity forces in order to assemble the $B$-elements from infinity, is given by the self-potential energy $\Omega_B$, while the work $L_t$ done by the tidal gravity forces in order to put the $B$ component together with the $D$ one from infinity through all the tidal distortions, is given by the tidal potential energy $V_{BD}$. Then a small variation $\delta V_B$ for a small displacement $\delta \vec{r}_B$ of all $B$ points, when $D$ is frozen (see LS5), writes (see, Sec.(1.7)):

$$\delta V_B \simeq \delta L_s + \delta L_t ; \quad (1.26)$$

In other words if $B$ contracts, less $DM$ lies inside the $\Sigma^*$ surface and the self gravity increases. The opposite occurs if $B$ expands itself. Therefore, even if both forces are attractive, the works for a virtual displacement are of opposite signs (see LS5).
1.3 Scaling relations at the special configuration

The physical explanation for the main features of the FP for ETGs, and in general for all virialized structures, is still an open question.

According to the dynamical theory here briefly explained the tilt can be explained by assuming a strict homology which does not necessarily imply a constant $M/L$ ratio; the scale length induced on the gravitational baryonic field by the $DM$ halo distribution is the interpretation key.

**Outputs vs. observables**

The mechanical property of the tidal configuration to distribute in about equal parts self and tidal energies, yields some outputs which are in good agreement with observations.

From Eq.(1.25) we have:

$$A = \frac{2}{2 - d}; \quad (1.27)$$

and taking into account the two relationships which connect the FP coefficients $A$ and $B$ with the observed FP (Djorgovski & Santiago, 1993):

$$\begin{align*}
A &= \frac{2(1 - \alpha_t)}{1 + \alpha_t}; \\
B &= -\frac{1}{1 + \alpha_t};
\end{align*} \quad (1.28)$$

we immediately obtain the tilt of the FP ($M/L \sim M^{\alpha_t}$):

$$\alpha_t = \frac{1 - d}{3 - d}; \quad (1.29)$$

and the other coefficient:

$$B = -\frac{3 - d}{2(2 - d)}; \quad (1.30)$$

as a function of the $DM$ distribution. The surprising result is that the quantities $A$, $B$, $\alpha_t$ which define the FP and its tilt are independent of $m = M_D/M_B$; this implies that the ETGs which belong to the FP may have different fractions of baryonic to DM mass, but their DM density profile must be the same.
If we choose, e.g., \( d = 0.5 \), we obtain \( A = 1.33 \), \( B = -0.83 \) and \( \alpha_t = 0.20 \) in good agreement with observations in the \( B \) band. In general we have:

\[
d = 0 \div 1 \implies A = 1 \div 2 \quad - B = 0.75 \div 1 \quad \alpha_t = 0.33 \div 0 \quad ; \quad (1.31)
\]

in good agreement with the data shown in Tab.2.2 for all bands. We can see that the variation interval of \( A \) is significantly larger (\( \Delta A = 4 \Delta B \)) than that of \( B \), in agreement with observations.

This allows us to expect that for all galaxies with DM halos similar to those of ETGs (e.g., spiral galaxies), a FP with similar \( A \), \( B \) and \( \alpha_t \) exponents must exist, no matter the luminosity distribution shape. This is in good agreement with the *cosmic meta-plane* of Burstein et al. (1997).

### 1.3.1 Cosmological framework

The exponents \( A \), \( B \) and \( \alpha_t \) which define the FP as a whole, are not directly linked with past cosmological conditions. This confirms what Djorgovski...
1.3. SCALING RELATIONS AT THE SPECIAL CONFIGURATION

(1992) already noted for scaling relations in a CDM scenario. In fact, translating the exponents $A$ and $B$ in cosmological quantities, the FP condition becomes:

$$2n_{\text{rec}} + 10 = A(1 - n_{\text{rec}}) - B(12\alpha_t + 4n_{\text{rec}} + 8) \; ;$$

(1.32)

where $n_{\text{rec}}$ is the effective spectral index of perturbations at the recombination epoch. Combining this last equation with Eqs.(1.27, 1.29, 1.30) we obtain an identity so that every information on $n_{\text{rec}}$ is lost. On the other hand in the projections of the FP on the coordinate planes, the dependence on the cosmological spectral index appears via the parameter $\gamma'$:

$$\frac{1}{\gamma'(M)} = \frac{1 + 3\alpha_{\text{rec}}(M)}{3} = \frac{5 + n_{\text{rec}}}{6} \; ;$$

(1.33)

where, according to Gott & Rees (1975) and Coles & Lucchin (1995)(Chapts. 14, 15), $\alpha_{\text{rec}}$ is the local slope of the CDM mass variance $\sigma^2_M$, at recombination time $t_{\text{rec}}$, given by:

$$\alpha_{\text{rec}} = -\frac{d \ln \sigma_M(t_{\text{rec}})}{d \ln M} \; ;$$

(1.34)

The dependence of the local slope $\alpha_{\text{rec}}$ on the DM halo mass does not significantly change in the $\Lambda$CDM scenario (to be tested).

If the total energy is conserved from the maximum expansion phase to virialization (with an energy re-distribution due to the violent relaxation mechanism), the three main quantities of FP have the following dependences on $m$ and $M_B$:

$$r_e \sim m^r M_B^R \; ; \quad r = \frac{(3 - d) - \gamma'}{\gamma'(3 - d)} \; ; \quad R = 1/\gamma' \; ;$$

(1.35)

$$I_e \sim m^i M_B^I \; ; \quad I = i = 2\frac{\gamma' - (3 - d)}{\gamma'(3 - d)} \; ;$$

(1.36)

$$\sigma_o \sim m^s M_B^S \; ; \quad s = -\frac{1}{2} \frac{(3 - d) - \gamma'}{\gamma'(3 - d)} \; ; \quad S = \frac{1}{2} \frac{\gamma' - 1}{\gamma'} \; ;$$

(1.37)

not only directly related to the DM distribution, via the exponent $d$, but also to the perturbation spectrum via $\gamma'$.

It is now possible to write the FJ relation in general form:

$$L \sim m^2 \frac{(3 - d) - \gamma'}{(3 - d)^2(\gamma' - 1)} \sigma_o^{\gamma'/(\gamma' - 1)(3 - d)} \; ;$$

(1.38)
For a typical DM halo of $M_D \simeq 10^{11} M_\odot$, we have $\gamma' \simeq 2$ Gunn (1987), and so we obtain:

$$L \sim m_j^{(d)} \sigma_o^{J(d)} ;$$  \hspace{1cm} (1.39)

a result in surprising accordance with observational data, indeed:

$$d = 0 \div 1 \implies \begin{cases} j = 0.44 \div 0 \\
J = 2.7 \div 4 \end{cases}$$

and for a value of $d = 0.5$ we obtain:

$$L \sim m^{0.16} \sigma_o^{3.2}$$  \hspace{1cm} (1.40)

The range results are in good agreement with Faber et al. (1989) $(J = 2.61 \pm 0.08$ in $B$-band) and with the significantly steeper slope $J = 4.14 \pm 0.22$ in $K$-band by Pahre et al. (1998a). The theory predicts an intrinsic different scatter in every FJ relationship due to the role of the factor $m^j$ (it is wider as $j$ increases, and disappears only when $d = 1$, i.e. $\alpha_t = 0$ so the tilt of the FP disappears, see SubSec.(1.3.2)).

It is also interesting to note that $\langle I \rangle_e$ decreases as $M_B$ increases as soon $d < 1 \ (3 - d > \gamma')$; in fact we have:

$$L \sim M_B^{0.8} ; \ r_e \sim M_B^{0.5} \ \Rightarrow \ \langle I \rangle_e = L/2\pi r_e^2 \sim M_B^{-0.2} ;$$  \hspace{1cm} (1.41)

At the Clausius’ minimum configuration, the following holds:

$$(M_{tot})_{at} = M_B \left(1 + \frac{\nu_{MB}}{\nu'_{\gamma}(2 - d)} \right) ;$$  \hspace{1cm} (1.42)

where $(M_{tot})_{at}$ is the total mass $M_{tot}$ at the tidal configuration $a_t$. If the density profiles of $B$ and $D$ component are universal, the following also holds:

$$L/(M_{tot})_{at} \sim L/M_B ;$$  \hspace{1cm} (1.43)

so that the two ratios are simply proportional. Cappellari et al. (2006) found $M/L \sim \sigma_o^{0.8}$ for an observed sample of either fast rotators or non-rotating ETGs and S0 galaxies. In our approach, we obtain:

$$\frac{M_B}{L} \sim m^{\alpha_1 \frac{(3-d)-\gamma'}{(3-d)(\gamma'_1-1)} \sigma_o^{2\gamma' - \gamma'(M_D)^{-1}} ;$$  \hspace{1cm} (1.44)
1.3. SCALING RELATIONS AT THE SPECIAL CONFIGURATION

On the same $DM$ scale we have an exponent 0.8 for $\sigma_o$ and a negligible dependence on $m$ (being $\sim m^{0.04}$) in perfect agreement with Cappellari et al. (2006) but also with Jørgensen (1999) value of $0.76 \pm 0.08$. At $a_t$ the following also holds:

$$\log \left( \frac{\overline{M}_D}{M_{tot}} \right)_{a_t} = -\log \left[ 1 + \frac{\nu'_V}{\nu_{Q_B}} (2 - d) \right] ;$$

meaning that $\left( \frac{\overline{M}_D}{M_{tot}} \right)_{a_t}$ only depends on the luminous and $DM$ density profiles; if they are both universal for the galaxy family considered, this $DM$ fraction has to be the same for all members (i.e. not depending on $m$). If the probable value for $d$ is around 0.5 and $b$ ranges from 2 ÷ 3 (e.g., Jaffe, 1983; Hernquist, 1990), we obtain $\log \left( \frac{\overline{M}_D}{M_B} \right)_{a_t} = 0.37 \div 0.69$, where $\log \left( \frac{\overline{M}_D}{M_B} \right)_{a_t} = 0.50$ at $b = 2.5$. The agreement with the histogram in Fig.5 of Jørgensen (1999), related to ETGs in the central part of the Coma cluster when the same IMF is assumed, appears to be very good.

1.3.2 About the tilt

In order to obtain a tilt of the FP we need to have a maximum in Clausius’ virial energy, that requires to have the equipartition between self and tidal energy of the $B$ component.

By considering the derivative of Clausius’ virial with respect to $a_B$, with the constraints given by Eq.(1.10) and Eq.(1.12), we conclude that the $D$ mass has to increase steeper than $(a_B/a_D)$ in order to obtain the tilt. This means that $\rho_D$ has to decrease less than $1/r^2$ at the border of the $B$ component, so that tidal energy may overcome self energy from this border forwards. But we can go further deducing a stricter constraint.

Let’s consider Eq.(1.36): $\langle I \rangle_e$ depends on the cosmological history of the galaxies, on $DM$ total mass and mass distribution and on the $B$ component mass. But the $L/M_B$ ratio (i.e., the FP tilt), is totally independent on the cosmic perturbation spectra and on the mass ratio $m$ and turns out to depend only on the $DM$ density profile. In fact, being:

$$L \sim I_e R_e^2 \implies m^{i+2r} M_B^{i+2r}$$

(1.46)
we have:

\[
\frac{L}{M_B} \sim m^{i+2r} M_B^{l+2R-1}
\]

(1.47)

where the exponent \(i + 2r = 0\) shows the lost connection of the tilt with \(m\) and the exponent \(I + 2R = 2/(3 - d)\) shows the lost connection of the tilt with cosmology (through \(\gamma'\)).

Moreover in order to have a positive tilt (as observed), we need:

\[
2/(3 - d) < 1 \implies 0 < d < 1;
\]

(1.48)

Therefore, the slope of the FP tells us a constraint on the density distribution of \(DM\) halos. To have a positive tilt we need the \(DM\) mass to increase steeper than \((a_B/a_D)^2\) at the border of the \(B\) mass; this means, in turn, that \(\rho_D\) has to decrease less than \(1/r\).

### 1.4 Transition toward virialization

How the special configuration characterized by the CV maximum may be reached, during the stellar system evolution? The answer is strictly connected with the problem of the end state of the collisionless stellar system after a violent relaxation phase (Lynden-Bell, 1967; van Albada, 1982; Binney & Tremaine, 1987 (Chapter 4); Burkert, 1994; White, 1996) and then to the problem of the constraints under which this phase occurs (Merritt, 1999). In consequence of this mechanism the proto-structure undergoes a sequence of contractions and expansions during which Landau damping ensures the buildup of a gradually increasing random kinetic energy due to the conversion of radial ordered velocity into a dispersion velocity field (see, e.g., Huss et al., 1999). But due to the virial equation, maximum of CV means minimum of the macroscopic pressure (that is minimum value of \(T_B\)), the stellar system needs in order to virialize. This makes stronger the idea to look at this special configuration as the best candidate for the initial virial stage because it has the least requests for sustaining the structure in virial equilibrium and it could also justify why the ellipticals are not completely relaxed systems in respect to the collisionless dark halo, the problem addressed by White & Narayan (1987). Moreover, the thermodynamical approach may help us to understand when and why the stellar system choose to virialize on the CV maximum.
1.5 The thermodynamical linear approach

Since Lynden-Bell (1967) first attempt to derive a statistical theory according to the Vlasov equation, the thermal equilibrium in collisionless systems after violent relaxation is still an open problem.

1.5.1 The temperature problem

We will enter deeply in this very complicate matter in the next chapter. Here we limit ourselves to adress the problem as follows:

1. the time derivative of the gravitational potential, which is the engine of the violent relaxation mechanism, is proportional to the energy per unit mass of a system’s star (Binney & Tremaine, Chap.4, 1987);

2. to avoid the segregation of mass, not observed in ellipticals, we have to remove the velocity dispersion problem when a collisionless system of different mass populations is considered. Actually, in the Lynden-Bell statistics the equilibrium distribution becomes a superposition of Gaussian components with different velocity dispersion in the non-degenerate limit;

3. after the Shu (1978) approach and the statistical attempts of Kull et al. (1997), the problem has been resolved by Nakamura (2000). At fixed value for mass, energy and phase-space volume of the system, Nakamura obtains a single Gaussian distribution for the equilibrium state as soon as a smooth initial fine-grained distribution function (DF) is assumed. Then the same mean dispersion velocity \( \langle \sigma^2 \rangle \) has to characterize the different mass populations. That means also the same energy per unit mass regardless from the stellar mass considered according to the typical relaxation process with no mass segregation (item 1);

4. we will assume as an Ansatz partially justified by Nakamura’s result (Nakamura, 2000) that for each stellar virial configuration of the \( B \) system a mean temperature is given by:

\[
T_S = \frac{m_* \langle \sigma^2 \rangle}{k}
\]
where $m_*$ is the mean mass of the stars and $k$ is the Boltzmann constant (see, e.g., Lima Neto et al. 1999; Bertin & Trenti, 2003).

### 1.5.2 Thermodynamic information

The knowledge of the mean temperature allows us to take into account the thermodynamic information, according to Layzer (1976):

$$I = S_{\text{max}} - S$$  \hspace{1cm} (1.50)

where $S_{\text{max}}$ means the maximum value the entropy of the system may have as soon as the constraints of it, which fix the actual value of its entropy to $S$, are relaxed. According to the $II^o$ Thermodynamic Principle it means that the state with $I \rightarrow 0$ is the natural attractor for a system.

Assuming as reference the entropy value which corresponds to the state of maximum possible volume for $B$ ($x = 1$), we are able to compare how changes the information at each configuration $x$ relative to this state:

$$I(x) = I(1) - V(x)$$ \hspace{1cm} (1.51)

$$S(x) - S(1) = V(x)$$ \hspace{1cm} (1.52)

Actually, we may move from the state $x = 1$ to the general state $x$ by a virtual sequence of thermodynamical quasi-static infinitesimal transformations (associated to an infinitesimal contraction $\Delta a_B < 0$) where the internal energy of the $B$ component is identified with the dominant peculiar kinetic energy of the stars, $T_B$, (which produces the macroscopic pressure of the $B$ subsystem) and the virial equilibrium is rearranged at each step (LS5). Under the assumption of a frozen dark component, the variation of the work done against the pressure by both the active forces on the $B$ system, self- and DM-gravity, is given by $\Delta V_B$ according to the result of Sec.(1.7). Then the heat variation $\Delta Q$ the structure is able to exchange with the surrounding medium is:

$$\Delta Q = \frac{1}{2} \Delta V_B$$ \hspace{1cm} (1.53)

In turn, $T_B \simeq \frac{1}{2} N m_* < \sigma^2$, then the mean temperature (1.49) becomes:

$$T_S = \frac{-V_B}{Nk}$$ \hspace{1cm} (1.54)

$N$ being the star number. Then virial equilibrium yields that the variation of the internal energy of the $B$ system, $\Delta T_B$, during the transition between
two virial states due to a small contraction, $\Delta x$, goes: one half to increase
the mean temperature of $B$, $\Delta T_s \sim -\frac{1}{2} \Delta V_B$, the other half has to be lost
by radiation outwards (Eq.(1.53)). The result is formally the classical one
found by Chandrasekhar (1939) and Schwarzschild (1958) for a single gaseous
component with the microscopic internal energy due to the molecular motion.
But now, the variation of CV energy which gives the work done by both the
gravitational forces actives on the $B$ system, is a non-monotonic function of
$x$ (Fig.1.2).

As pointed out in LS5 in this last case we cannot reach equipartition
starting from $x = 1$ by a simple contraction of the system considered as
isolated but we need of a supplement of heat source without which the CVM
configuration is missed.

The entropy (normalized to the factor $Nk/2$) of the $B$ system at $x$ in
respect to that at $x = 1 , \text{ is given by (LS5):}

$$\tilde{S}(x) - \tilde{S}(1) = \ln \frac{V_B(1)}{V_B(x)} = V_n(x)$$

(1.55)

The trend of $I(x)$, Eq.(1.51), in arbitrary units, is shown in the Fig.1.3
where the minimum value of the information reached inside the sequence is
arbitrarily put equal to 0.

The corresponding trend of $V_n(x)$ is plotted in Fig.(1.4)

It should be noted that the configurations corresponding to the CV maxima,
are all characterized by the property to exhibit the maximum value of
$V_n(x)$, that means the maximum of the entropy related to the state $x = 1$
inside the entropy spread of the other configurations. According to the $II^o$
Thermodynamical Principle, in these linear models the CV maximum appears as the most natural configuration for the baryonic component. In this
sense we may use the following expression: the CV maximum becomes a
virialization attractor.

But this is a common feature of maxima only in the linear approximation
models; it may fail in the general non-linear case. As we will see afterward,
the common property of the CV maxima in both linear and non-linear cases
is that their capability to become virialization attractors is strictly connected
to the value of the slope the DM density profile has at $x_t$ and their attractivity
increases as the slope (in absolute value) decreases. In other words, in order
the $B$ component settles into the CV maximum configuration at the end of
relaxation, the CV maximum has not only to exist but it has to fall in a way
that the main bulk of stellar component has to be located inside a DM halo in which the density doesn’t decrease faster than a given threshold.

1.6 CV maximum as attractor

How is possible to quantify the *attractivity* of CV maximum? According to considerations and definitions given in the previous section, this role may be assumed by $V_n(x_t)$. In the linear model and in the limit of null inner cores, the Eq.(1.55) at $x_t$ becomes a completely analytical function with slopes $d,b$ and mass ratio $m$. That means, for any value of the slope $b$ and of parameter $m$, provided over the threshold which produces the CV maximum, the relative
1.6. CV MAXIMUM AS ATTRACTOR

Figure 1.4: Trends of the entropy function $V_n(x)$ (eq.(1.55)) for the $B$-system, in arbitrary units (see text), as function of $x$. The cases are corresponding to Tab.2.1. The maxima occur at the tidal radii (Raffaele, 2003).

entropy at $x_t$ is:

$$\tilde{S}(x_t) - \tilde{S}(1) = \ln \frac{V_B(1)}{V_B(x_t)} = V_n(x_t, b, d, m); \quad (1.56)$$

which exhibits the following features.

Assigned $\bar{b}$ and $\bar{m}$, the trend of the function $V_n(x_t, \bar{b}, d, \bar{m})$ exhibits always an absolute maximum at $d = 0$, for fixed $x_t$, $b$ and $m$. Normalized to this maximum value the function (1.56) is plotted in the cases: $b = 1, m = 10$ (upper curve) and $b = 1, m = 5$ (lower curve) Fig.(1.5). It should be noted that by passing from $d = 0$ to $d = 0.5$ in the upper and lower curve, at the CV maximum the relative entropy remains at about the 87% ÷ 85%, respectively, of the maximum amount of available entropy. When $d$ rises over 1 it drops from the 67% ÷ 61%, respectively, downwards to zero. But, what is the weighted mean of the $d$ absolute slope one has to expect from
Figure 1.5: Trend of the function $V_n(x_t, \bar{b}, d, \bar{m})$ normalized to its maximum value at $d = 0$ (Eq.(1.57)) in the cases: $b = 1$, $m = 10$ (upper curve) and $b = 1$, $m = 5$ (lower curve).

the distribution of Fig.(1.5)? If we perform the weighted mean $d_{wm}$ given to each value of $d$ as weight the corresponding value of the normalized function:

$$\tilde{V}_n(x_t, \bar{b}, d, \bar{m}) = \frac{V_n(x_t, \bar{b}, d, \bar{m})}{V_n(x_t, \bar{b}, d = 0, \bar{m})} \quad (1.57)$$

That means:

$$d_{wm} = \frac{\int d \cdot \tilde{V}_n(x_t, \bar{b}, d, \bar{m}) \, dd}{\int \tilde{V}_n(x_t, \bar{b}, d, \bar{m}) \, dd} \quad (1.58)$$

It is noteworthy that it results to be, $d_{wm} = 0.4681$, in the case $m = 10$ and $d_{wm} = 0.4618$ in the case $m = 5$. That gives deep meaning at the value
1.7. SMALL DEPARTURES FROM VIRIAL EQUILIBRIUM

$d = 0.5$ used in the whole linear theory in order to match the theory with many FP observations.

The general message which comes from Fig.1.5 is that in order to increase the attractivity of a CV maximum, expressed by Eq.(1.56), we need to decrease the slope at which the maximum falls. We will come back to this conspiracy between the *attractivity* and the slope in the non-linear thermodynamical approach considered in the chapt.6.

1.7 Small departures from virial equilibrium

We will consider now what is the mathematical form of the $V_B$ potential energy variation as soon as the $B$ inner system contracts or expands its initial volume $S_o$ of a small quantity $\Delta S_o$. Following Chandrasekhar’s analysis (Chandrasekhar, 1969, Chapter 2): by definition the Clausius virial is a global, integral parameter of an *extrinsic* attribute, that means of a quantity which is not intrinsic to the fluid element (like pressure or density) but something, we name $F(\bar{x})$, which it assumes simply by virtue of its location as the gravitational potential and its first derivative are. Then the variation of the integral:

$$
\delta \int_{S_o} \rho_B F \, d\bar{x}_B = \int_{S_o + \Delta S_o} \rho_B F \, d\bar{x}_B - \int_{S_o} \rho_B F \, d\bar{x}_B
$$

(1.59)

when fluid volume changes from $S_o$ to $S_o + \Delta S_o$ by subjecting its boundary to the displacement $\xi(\bar{x}, t) = \bar{x} - \bar{x}_o$, may be transformed into the integral over the unperturbed volume, that is:

$$
\delta \int_{S_o} \rho_B F \, d\bar{x}_B = \int_{S_o} \rho_B \Delta F \, d\bar{x}_B
$$

(1.60)

where $\Delta F$ is the Lagrangian change in $F$ consequent to the displacement $\xi$.

The extension of this analysis to two-component systems has been performed with the following result:

$$
\delta \Omega_B = - \int_{S_o} \rho_B \sum_{r=1}^{3} \xi_r \frac{\partial \Phi_B}{\partial x_r} \, d\bar{x}_B
$$

(1.61)

$$
\delta V_{BD} = - \int_{S_o} \rho_B \sum_{k=1}^{3} \sum_{r=1}^{3} \xi_k \frac{\partial}{\partial x_k} \left( x_r \frac{\partial \Phi_B}{\partial x_r} \right) \, d\bar{x}_B - \int_{M_o} \rho_D \sum_{r=1}^{3} \xi'_r \frac{\partial \Phi_B}{\partial x_r} \, d\bar{x}_D
$$

$$
+ \int_{M_o} \rho_D \sum_{k=1}^{3} \sum_{r=1}^{3} \xi'_k \frac{\partial}{\partial x_k} \left( x_r \frac{\partial \Phi_B}{\partial x_r} \right) \, d\bar{x}_D
$$

(1.62)
where the unperturbed volume of $D$-component is $M_0$ and $\vec{\xi}'$ is the amount of the perturbation in the point domain of the same component.

Under the assumption of a frozen dark component, $\vec{\xi}'$ turns out to vanish and Eq.(1.62) reduces to:

$$
\delta V_{BD} \simeq \delta L_s + \delta L_t \simeq - \int_{S_0} \rho_B \sum_{r=1}^{3} \xi_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_B
$$

$$
- \int_{S_0} \rho_B \sum_{k=1}^{3} \sum_{r=1}^{3} \xi_k \frac{\partial}{\partial x_k} (x_r \frac{\partial \Phi_D}{\partial x_r}) \, d\vec{x}_B = (\delta V_B)_{a_D}
$$

By definition of tidal radius, which is the $B$ dimension at the maximum of its Clausius virial energy, at frozen $a_D$, the $(\delta V_B)_{a_D}$ is stationary at $a_t$ (Fig.1.2)(see, LS1); then by moving of a virtual\(^2\) displacement $\delta a_B$, from $a_B = a_t$, we have:

$$
\delta L_s + \delta L_t \simeq (\delta V_B(a_t))_{a_D} = 0
$$

This means the configuration at $a_t$ satisfies the d’Alembert Principle of the virtual works (see also LS1). The physical reason is the following: if, e.g., $B$ contracts, less dark matter enters inside the $S_B$ surface, in the meanwhile the self gravity increases. The opposite occurs if $B$ expands itself. Then, even if both forces are attractive, the works which correspond to them, for a virtual displacement, are of opposite signs (see, LS1). Then the tidal radius configuration is an equilibrium configuration even if not stable because the total potential energy of $B$ has not a minimum (Fig.1.2).

\(^2\)Here virtual means at frozen D.
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